Ranges and Behaviour of Rifled Projectiles in Air.

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In a former paper it was shown that the distance which a pointed projectile* would travel in air, when air resistance was the only force acting on it, could be very closely expressed by the formula $s = v't + \frac{u_0}{a}(1-e^{-at})$. In the present paper I propose to examine the nature of the motion of such a projectile when gravity acts on it, as well as air resistance.

Suppose that projectile is started at an angle a with the horizon. Without the action of gravity at the end of time t, the projectile will have attained a height of $s \sin a$ above the surface of the ground. It is easy to see (the proof, however, is given in the notes) that, if gravity acts and is the only force which has a component normal to the trajectory, the position of the projectile at t will, if x and y are the horizontal and vertical co-ordinates, be $x = s\cos a$, $y = s\sin a - \frac{1}{2}gt^2$. Thus if we put $y = 0$, *i.e.*, if the projectile reaches the ground after time t, the range for an elevation α is $s = \frac{1}{2}gt^2/\sin a$, so that $\sin a = gt^2/2s$. Experiment, however, shows that this relation does not hold, and it must be concluded, therefore, that gravity is not the only force acting which has a component normal to the trajectory.

Experiment also shows that the axis of an elongated rifled projectile tends to set itself in the direction of the tangent to the trajectory, and it will appear that the small angle between the axis and the tangent is the origin of the normal force which causes the simple relation $\sin a = gt^2/2s$ to fail; the fact being that the long projectile with its axis slightly inclined to the direction of motion behaves as a rather imperfect sort of flying machine, the upward component of the resistance on the underside of the shot acting to diminish the effect of gravity.

That this was so was well known to the late W. E. Metford, who did so much for the improvement of small arms; but I cannot find that it has been taken into account by artillerists, nor, as far as I know, has any dynamical explanation of the tendency of the axis of a rifled shot to follow the direction of the tangent to the trajectory been published, though part, at any rate, of the true reason was given in conversation by the late W. Froude more than 30 years ago.

* The projectile here referred to is ogival-headed, the radius of the ogive being two diameters.

We know that if a rifled shot were fired *in vacuo* its axis of rotation would remain parallel to its initial position throughout its flight. In order to examine the reason why the same shot fired in air should keep its axis of rotation nearly parallel to the tangent of the trajectory, it will be convenient to take at first a simpler case.

Imagine a top or gyrostat of any shape (any surface of revolution) supported in gymbals and placed at one end of a long arm, in a vertical plane, the other end of the arm being fixed to a horizontal axis (see fig. 1).

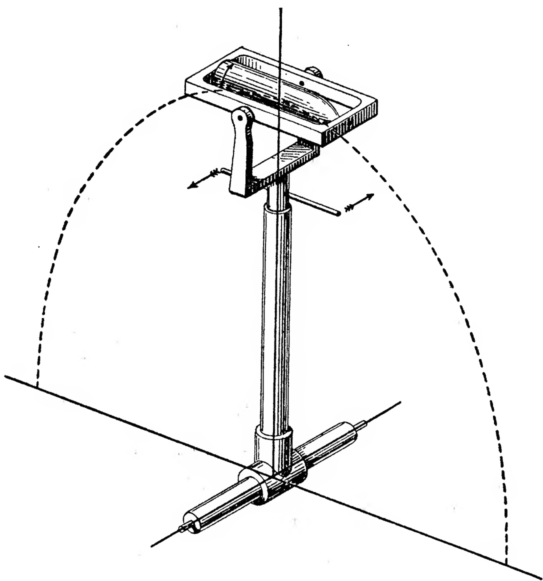

FIG. 1.

If the gyrostat be set spinning, and the arm is then made to rotate about its horizontal axis, we know that, in the absence of any external force, the axis of the gyroscope will always remain parallel to the position it occupied when first set spinning.

Suppose this direction to be the tangent to the circle described by the C.G. of the shot about the horizontal axis AB, I will first inquire what the magnitude and direction of the couple must be which will cause the axis of the shot to remain tangential to the circle when the arm revolves at a definite angular speed.

The mathematical expressions for the required couple are given in the notes.

It appears (a) that the axis of the requisite couple is parallel to the arm R; (b) that it must tend to cause rotation in the same direction as the projectile would be spinning if the head of the latter was pointing towards the axis AB;

(*c*) that its magnitude is directly proportional to the angular speed of the arm about AB and to the angular speed of the gyrostat about its axis; and (*d*) to the fourth power of the linear dimensions of the projectile.

If, now, in imagination, we make the arm R equal in length to the radius of curvature of the trajectory of a projectile and the angular speed of the arm such that its end has the speed of the projectile, and give the gyrostat the spin which would be given by the rifling, we know that, in some way or other, the air must call into action a couple with the above specified direction and magnitude.

As a numerical example showing the order of magnitude of the required couple, I give the following figures relating to a 12-inch, 6-inch, and 0·303-inch projectile. The couple which must act about the radius of curvature of the trajectory through the C.G. of a projectile having a velocity of 2000 f.s. and making one whole turn about its axis while it travels 30 diameters is:—

12-inch gun	11·15 ft.-lbs.
6-inch gun	0·66 „
0·303-inch rifle	$1·02 \times 10^{-5}$ ft.-lb. = about 8·6-inch grains.

The action of the air in producing this couple may best be illustrated by the action of a jet of water on a rotating body. It is evident that if the axis of the jet and the axis of rotation are identical, the rotation communicated to the fluid by friction, as it spreads over the surface of the body after impact, will be symmetrical about the axis, and the speed of rotation of the projectile will, therefore, unless maintained from without, gradually decrease, but beyond this there will be no effect. The case is different, however, if the axes of the jet and of rotation are inclined.

Suppose in the first place that these two axes intersect at a point in the projectile (which may be called the centre of pressure) independent of the angle (provided the angle remains small), and that the mass in the projectile is so distributed fore and aft as to make this point the centre of gravity. In this condition the jet, although it impinges obliquely on the projectile, exerts no couple in the plane containing the two axes.

There is a couple, however, acting in a plane through the jet perpendicular to the plane containing the axis of the jet and the axis of rotation; for the jet impinges on the surface of the projectile at some distance from the axis of rotation. Surface friction therefore imparts sideways velocity to the jet after impact. Thus the reaction constitutes a force urging the surface of the projectile, at the point of impact, in a direction opposite to that in which the surface is moving, and this force can be calculated. If the effect on

impact were the only action of the jet, the force just mentioned, together with the inertia of the shot, would constitute a couple acting in the direction required to bring the axis of rotation towards the axis of the jet.* As a matter of fact, however, the fluid in leaving the surface of the projectile also exercises a force there, the resultant of which is in the opposite direction to that called up by the impact of the jet.

The magnitude of this force cannot at present be calculated from dynamical principles, but can be shown to exist by experiment, and by appropriate experiments could be measured.

Its origin can be explained thus: Suppose a cylinder immersed in a current of fluid with its axis at right angles to the stream. If the fluid were the perfect fluid of mathematicians, the stream lines would flow in behind the cylinder in curves precisely similar to those they followed while approaching it.

A real fluid, such as air or water, does not behave in this way. In approaching the solid the stream lines conform nearly to the theoretical flow of a perfect fluid, but, instead of closing in behind, the streams leave the solid surface altogether near its greatest diameter, and enclose between them a body of fluid constituting a wake. The motion of the fluid in the wake is of a very complicated character, consisting of eddies which are always being formed in the immediate neighbourhood of the solid, and breaking away from it when they have reached a certain size.

The angle which the average of the wake makes with the direction of flow is unstable within certain limits, so that the wake tends to trail away from the solid at some small angle to the general direction of the stream, implying a lateral force on the solid.

If the cylinder is stationary, the direction which the instability takes varies quickly, and, looking back along the wake for some distance, its mean direction will appear to be that of the stream, but its course will be sinuous, owing to the successive changes in the direction of instability. If, however, the cylinder is made to revolve, the direction of the instability is settled, and the wake leaves with a sideways component of velocity in the direction of the motion of the rear surface of the cylinder.

Even a slow rotation of the cylinder (*i.e.*, a rotation giving a surface velocity less than the velocity of the stream) suffices to give the wake a considerable lateral deviation; but the angle between the wake and the stream increases with the velocity of rotation of the cylinder, though in what proportion I do not at present know (the curious flight of a golf ball with underspin depends on the action just described).

* This was the explanation given by W. Froude.

The cylinder has hitherto been supposed to have its axis at right angles to the direction of flow, but now let it be gradually turned so as to bring the axis of rotation towards the direction of the stream. When the two are nearly coincident the greater part of the wake, of course, belongs to the base of the cylinder, but part still belongs to that part of the side which is, as it were, in shadow; and as long as any shadowed part remains, some of the wake leaves with a sideways velocity having the direction of the surface motion of the shadowed part. This necessarily implies a force on the cylinder in the opposite direction.

What the angle between the axis of the projectile and its direction of motion is at which the shadow vanishes, I have not determined. It is certainly small, and varies with the shape of the head; with flat heads it is probably less than a degree.

The statements above depend only on experiments and observations I myself have made. I have no doubt about the existence of the forces, but cannot give their magnitude.

We are left, however, with the fact that a projectile rotating in air, with its axis nearly in the direction of motion, experiences a force in opposite directions at either end which together constitute: (*a*) a couple tending to bring the axis of rotation towards the direction of motion, and (*b*) a force (equal to the difference of the forces at the head and tail) tending to move the projectile bodily sideways.* This force is one element of drift.

So far I have supposed that the C.G. of the projectile was at the centre of pressure. In all real projectiles, however, the centre of pressure is, in general, some way in front of the C.G., and, as has often been pointed out, this produces a couple in the plane containing the axis of rotation and the tangent to the trajectory whose magnitude is proportional to the angle between the two, and to the distance between the centre of pressure and the centre of gravity, that is to OC by OCA (fig. 2).

The effect of this couple on the spinning projectile is to cause its axis to describe a cone round the direction of the motion. The time, T, required for the axis to make one complete turn about the direction of motion (the precessional period) is given in the notes.

The air resistance also gives rise to a force tending to move the projectile bodily sideways (in the plane containing the axis of rotation and the direction of motion) which is directly proportional to the angle between them and to the resistance. (The horizontal component of this force is the other element of drift.)

* If the views here expressed are correct, the axis of rotation of a spherical rifled shot should tend to follow the tangent to the trajectory.

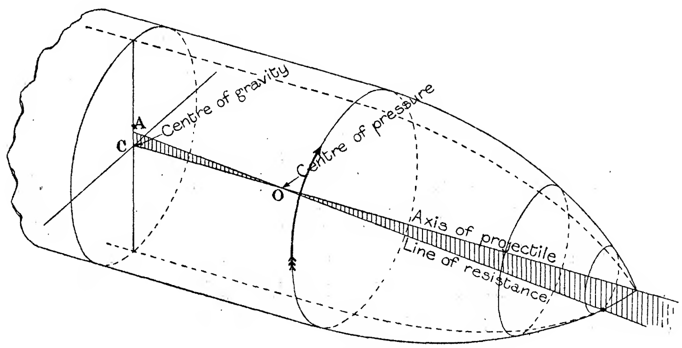

FIG. 2.

Now consider the trace of the axis of the projectile on a plane XOY (fig. 3), at right angles to the direction of motion, the plane having the same speed as the projectile, and O being the point where the tangent to the trajectory meets the plane.

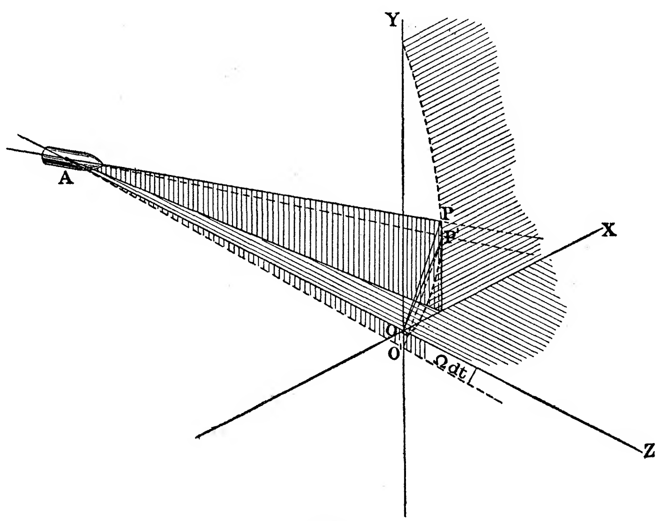

FIG. 3.

In virtue of the couple in the plane OAP($=C_1$), P would describe a circle about O with a velocity proportional to PO.

In virtue of the couple in the plane through Z at right angles to OAP ($=C_2$), P would describe the straight line PO with a velocity also proportional to PO. The combination of the two movements makes the trace of P into an equiangular spiral.

In a real trajectory the direction of the tangent is always changing.

Let O O′ (fig. 3) be the distance which the tangent would trace in time *dt*

on the plane XOY in virtue of the curvature of the trajectory (O O′ of course $= \mathrm{OA} \,.\, \Omega dt$). If the point P is now on the equiangular spiral at a position where the tangent to the spiral is parallel to the radius of curvature of the trajectory, the element P P′ of the spiral is equal to $2\pi \mathrm{T}^{-1} \mathrm{OA} \,.\, \sin\psi \cos\delta$, where $\psi = \mathrm{OAP}$, $\delta =$ constant angle of spiral, and $\mathrm{T} =$ precessional period, and if $2\pi\mathrm{T}^{-1} \sin\psi \cos\delta = \Omega$, ψ is independent of time, the axis of the shot in virtue of its precessional and inward motion changing its position at the same rate as the direction of the tangent to the trajectory.

With a well-formed shot the angle POY is small,* and this proves that the motion due to the couple C_2 (which may be called the extinctive couple, from the analogy of its effect with the effect of resistance to the motion of a conical pendulum) is large compared to the couple C_1.

If we take the algebraic sum of the horizontal and vertical components of the force at the C.G. of the projectile, the first produces drift and the second acts to diminish the effective force of gravity.

It appears, then, that the complete effect of the air resistance on the shot is to produce, in addition to the general retardation of its velocity, two couples: one in the plane containing the axis of the projectile and the tangent to the trajectory, and one in the plane through the tangent at right angles to the first plane; also two forces both normal to the direction of motion, parallel and perpendicular to the radius of curvature of the trajectory.

The reduction in the apparent force of gravity is very considerable and increases with the curvature of the trajectory, that is as the velocity of the projectile diminishes. The relation found between the lifting force and the apparent force of gravity is given in the notes, and is also shown in the diagram, fig. 4.

The meaning attached to "apparent value of gravity" in this paper is defined by the relation

$$\iint (g-f)\, dt = \tfrac{1}{2} g' t^2,$$

where f is the lifting component due to the obliquity of the axis of the shot and direction of its travel and g' is the "apparent value of gravity." In other words, g' is the uniform acceleration which, acting for a time t on a projectile started at an angle to the horizon for which the range is R, causes the projectile to fall through a distance $s \sin\alpha$. In diagram 4 the values of g' are the result of the analysis of a large number of range tables collected during many years. These tables, which refer to nearly all the calibres of rifled guns which have been in use, are based, for the most part, on the ballistic theories given in the text-books, but corrected by the use of

* If this were not so, the drift would be greater than it is.

suitable multipliers to make them agree with practice. It is clear from the diagram that the value of g' is fairly well expressed by the equation $g' = g - \frac{c}{v_0}(v_0 - v)$. For the purpose of finding a general formula for ranges this is all we require, and subject to the condition that the angle α is small enough for $\cos\alpha$ to be taken as unity, we can at once write:—

$$\mathrm{R} = v't + \frac{u_0}{a}(1 - e^{-at}). \quad \text{(A)}$$

$$a = \left(g - \frac{c}{u_0}(v_0 - v)\right)\frac{\mathrm{C}^2}{2\mathrm{R}}. \quad \text{(B)}$$

In the actual construction of range tables it is often convenient to plot R and a in terms of t and then take the simultaneous values, thus getting R in terms of α. There can be no doubt that these formulæ give the ranges with great accuracy when α is small, but a reference to the examples given at the end of this paper shows that even when α is as large as 10°, that is when $\cos\alpha$ differs from unity by nearly 2 per cent., the agreement is still very close.

The reason for this is that the difference between $\cos\alpha$ and unity, which should be taken into account in the range as given by equation (A) (in which the air is treated as being of constant density), is of the same sign and (with the larger guns) of nearly the same amount as the increase of range due to the diminished density of the air at the higher parts of the trajectory.

A graphic method of finding the correction for variable atmospheric density is given in the notes.

The object of this paper is to show that although experiments which will determine directly what the action of the air is on an

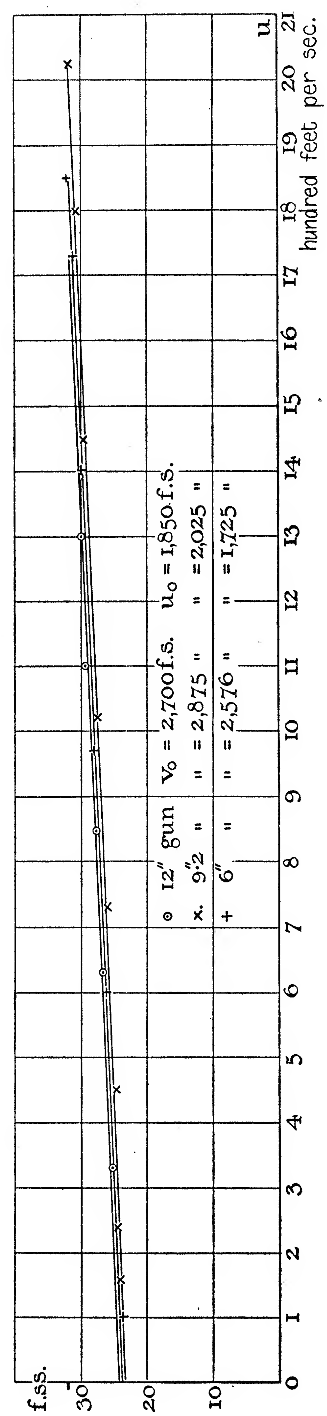

FIG. 4.

obliquely moving rifled projectile are very desirable, it is possible, nevertheless, even with existing knowledge to compute with sufficient accuracy the range of any projectile in terms of its initial velocity, weight and diameter, and without the use of any arbitrary constants, except such as are derived from resistance experiments and the constants in the expression for g'.

To be able to do this is an important matter, for it is not such a simple thing as might be supposed to get the true ranges of a large gun by direct experiment.

Range Tables, calculated by Formulæ (A) and (B) for 12-inch Gun, M.V. 2700 f.s.; and 6-inch Gun, M.V. 2575 f.s. Columns III and V give the differences between the Calculated Values and the Values given in the ordinary Range Tables for the same Guns.

12-inch gun.					6-inch gun.				
I. Range.	II. Elevation.	III.	IV. Time of flight.	V.	I. Range.	II. Elevation.	III.	IV. Time of flight.	V.
yds.	° ′	′	secs.		yds.	° ′	′	secs.	
0	0 ·0	0 ·0	0 ·0	0 ·0	0	0	0 ·0	0 ·0	0 ·0
500	11 ·5	+ 0 ·5	0 ·56	+0 ·00	500	14	+ 1 ·0	0 ·62	+0 ·02
1,000	26 ·0	+ 2 ·0	1 ·2	0 ·06	1,000	29	+ 2 ·0	1 ·22	−0 ·04
1,500	35 ·5	− 0 ·5	1 ·81	0 ·07	1,500	44	0 ·0	1 ·95	−0 ·02
2,000	49 ·0	0 ·0	2 ·41	0 ·07	2,000	1 0	− 1 ·0	2 ·72	+0 ·06
2,500	1 3 ·5	− 1 ·0	2 ·99	0 ·01	2,500	1 18	− 1 ·0	3 ·50	+0 ·06
3,000	1 17 ·0	0 ·0	3 ·69	0 ·06	3,000	1 38	− 1 ·0	4 ·38	+0 ·10
3,500	1 31 ·0	0 ·0	4 ·32	0 ·00	3,500	2 0	0 ·0	5 ·26	+0 ·08
4,000	1 47 ·0	− 1 ·0	5 ·02	0 ·02	4,000	2 26	+ 2 ·0	6 ·20	+0 ·04
4,500	2 3 ·0	− 2 ·0	5 ·75	0 ·03	4,500	2 55	+ 1 ·0	7 ·20	+0 ·02
5,000	2 20 ·0	− 4 ·0	6 ·50	0 ·03	5,000	3 25	+ 1 ·0	8 ·35	+0 ·10
5,500	2 36 ·0	− 3 ·0	7 ·29	0 ·00	5,500	4 0	0 ·0	9 ·55	+0 ·1
6,000	2 55 ·0	− 2 ·0	8 ·05	0 ·03	6,000	4 36	+ 4 ·0	10 ·75	+0 ·05
6,500	3 13 ·0	− 1 ·0	8 ·79	−0 ·1	6,500	5 19	+ 3 ·0	12 ·00	+0 ·03
7,000	3 34 ·0	0 ·0	9 ·67	−0 ·15	7,000	6 2	+ 3 ·0	13 ·40	+0 ·04
7,500	3 56 ·0	− 1 ·0	10 ·50	−0 ·02	7,500	6 48	+ 3 ·0	14 ·90	+0 ·10
8,000	4 19 ·0	0 ·0	11 ·38	−0 ·11	8,000	7 36	0 ·0	15 ·80	+0 ·5
8,500	4 43 ·0	+ 0 ·0	12 ·40	−0 ·04	8,500	9 8	+37 ·0	17 ·50	−0 ·37
9,000	5 7 ·0	∓ 1 ·0	13 ·42	−0 ·02	9,000	10 9	+39 ·0	19 ·20	−0 ·29
9,500	5 33 ·0	− 0 ·0	14 ·45	+0 ·04	9,500	11 10	+42 ·0	20 ·95	−0 ·2
10,000	6 1 ·0	+ 2 ·0	15 ·50	+0 ·06	10,000	12 12	+32 ·0	22 ·8	−0 ·07
10,500	6 27 ·0	+ 1 ·0	16 ·55	+0 ·01	10,500	13 13	+21 ·0	24 ·65	+0 ·01
11,000	6 54 ·0	− 2 ·0	17 ·60	+0 ·01					
11,500	7 22 ·0	− 4 ·0	18 ·65	−0 ·06					
12,000	7 58 ·0	− 1 ·0	19 ·8	−0 ·06					
12,500	8 32 ·0	− 2 ·0	20 ·9	−0 ·14					
13,000	9 14 ·0	− 9 ·0	22 ·2	−0 ·16					
13,500	9 54 ·0	−14 ·0	23 ·5	−0 ·17					
14,000	10 39 ·0	−23 ·0	24 ·9	−0 ·06					
14,500	11 23 ·0	−30 ·0	26 ·25	−0 ·05					

Computation of Part of Range Table for 12-inch Gun.

W = 850 lb. D = 1 ft. $v_0 = 2700$ f.s. $v' = 850$ f.s. $u_0 = 1850$ f.s. $a = (6{\cdot}385\ \mathrm{D}^2/\mathrm{W}) = 0{\cdot}784$. $v_0/a = 24{,}750$ ft.
$c = (8{\cdot}3/u_0) = 0{\cdot}0045$.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.	XIV.	XV.
x.	$t = \frac{x}{a}$.	e^{-x}.	$1-e^{-x}$.	$v't$.	$\frac{u_0}{a}(1-e^{-at})$.	S in feet $= v't + \frac{v_0}{a}(1-e^{-at})$.	$u = v_0e^{-at}$.	$v_0 - v$.	$c(u_0-u)$.	$g' = g - c(v_0-v)$.	$\frac{gt^2}{2}$.	$\sin\alpha = \frac{g't^2}{2\mathrm{S}}$.	α.	Range in yards.
0·4	5·34	0·6912	0·3288	4520	8,150	12,670	1241	609	2·73	29·47	422·5	0·0382	1° 54′	4233
0·5	6·66	0·6090	0·3910	5650	9,680	15,330	1128	722	3·26	28·94	641·5	0·0419	2 24	5110
0·6	8·00	0·5500	0·4500	6780	11,100	17,880	1019	831	3·75	28·45	924·0	0·0516	2 58	5960
0·7	9·36	0·4977	0·5023	7910	12,730	20,640	897	953	4·30	27·90	1229·0	0·0595	3 25	6880
0·8	10·65	0·4505	0·5495	9040	13,580	22,620	834	1016	4·56	27·66	1565·0	0·0699	3 59	7540

Plotting α in Terms of Range.

Range.	α.	Range.	α.
4000	1° 46′	6000	2° 59′
4500	2 3	6500	3 15
5000	2 20	7000	3 31
5500	2 39	7500	3 56

The accidental differences due to a variety of causes between the successive rounds of the same gun make it necessary that a large number of rounds should be fired in order that anything like a true mean should be arrived at, and in the case of large guns such as the 12-inch the alteration in the gun itself (whose life is only about 200 rounds) makes it almost impossible to determine by experiments on a single gun what the true range is when the gun is new. Any formula, therefore, which will give the range with accuracy is not only a matter of convenience, but of economy, both in ammunition and wear of guns.

In the examples given (p. 544) of angles of elevations and ranges computed by equations (A) and (B), the third and fifth columns give the difference between the computed angles and times of flight and those given in the most trustworthy of the ordinary range tables.

An example is also given (p. 545) of the actual computation of part of a range table for a 12-inch gun and it may be remarked that the whole of the work can be done without reference to any table except that for e^{-x} given in Column III.

NOTES.

Let X, Y, Z be principal axes of a solid of revolution, Z being the generating axis. Let the solid have an angular velocity ω about Z (fig. 5).

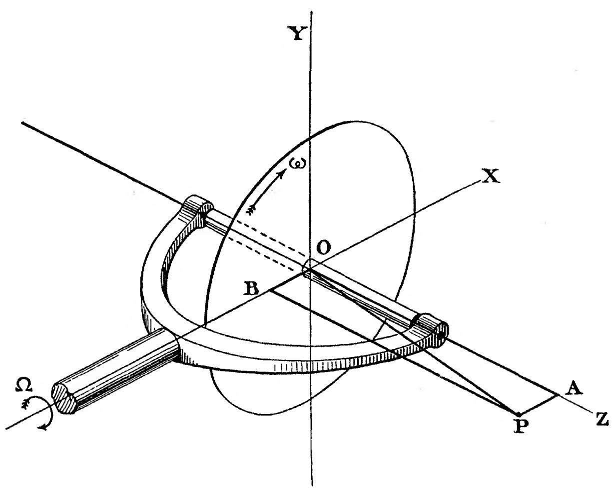

FIG. 5.

Now impose an angular velocity Ω about X, Ω/ω being small ($=$ OB/OA); then OP is the instantaneous axis of rotation, and OP (to the first degree of approximation) $=$ OA.

Imagine the solid compressed in the direction of Z until it becomes a circular disc in the plane XY, and let M be its mass and K its radius of gyration about a diameter.

The disc is revolving about OP with velocity ω. The components of the centrifuga

force of the disc parallel to Y exactly balance one another, but as regards the components parallel to X, since half the disc is on one side of the plane through O perpendicular to OP and half on the other, there will be a couple acting on the disc in the plane XZ, tending to make the plane of the disc coincide with the plane perpendicular to the instantaneous axis.

The magnitude of the couple is the integral of (components of centrifugal force parallel to X of all parts of the disc) $\times$ (their distance from Y) $\times$ (angle AOP), or

$$MK^2\omega^2\Omega/\omega = MK^2\omega\Omega = Q, \text{ say.}$$

This shows that if the gyrostat is made to turn with velocity Ω about X, the mechanical axis exerts a couple Q on its bearings about the axis of Y, *i.e.*, in the plane of XZ. Conversely, if a couple $-Q$ is made to act on the axis in this plane, the gyrostat will revolve with velocity Ω about X.

Suppose the axis of the couple to make the angles λ, μ, ν with X, Y, and Z respectively, the resolved couples about X, Y, and Z are $Q\cos\lambda$, $Q\cos\mu$, and $Q\cos\nu$. The effect of $Q\cos\nu$ will be merely to slightly alter the value of ω, and we need not further consider this component.

$Q\cos\lambda$ and $Q\cos\mu$ will each cause the mechanical axis to resolve with velocities $\Omega\cos\lambda$ and $\Omega\cos\mu$ in the planes XZ and YZ respectively.

Two particular cases may be noted with reference to a projectile :—

(*a*) When the plane of the couple is always in the plane containing the axis of symmetry and the instantaneous axis of the projectile, the axis of symmetry will describe a cone about the axis of the couple with velocity Ω.

(*b*) If the axis of the couple makes an angle $90° - \psi$ with the axis of symmetry of the projectile, these two axes will approach one another with the velocity $\Omega\sin\psi$ (or $\Omega\psi$ if ψ is small).

Case (*a*) corresponds to the action set up by the line of resistance not passing through the centre of gravity of the projectile, and case (*b*) to the action of the couple due to air friction and wake.

The precessional period T (or the time in which the mechanical axis describes a complete cone) is given by the following relations. If ρ is the radius of curvature of the trajectory and v the linear velocity of the projectile, $\rho\Omega^2 = g$. And since $\Omega T = 2\pi$ and $\rho\Omega = v$, we have

$$T = \frac{2\pi v}{g}, \quad (1) \qquad \Omega = \frac{g}{v}, \quad (2) \qquad \rho = \frac{v^2}{g}. \quad (3)$$

If we take ω, the angular velocity due to rifling, as being such that the shot makes one turn in n calibres, $\omega = \frac{2\pi v}{nD}$, hence the couple $Q\,(= MK^2\omega\Omega)$ becomes $\frac{W}{g}K^2\frac{2\pi}{nD}$; or, since K^2 for a circular disc $= \frac{1}{4}R^2$, and modern rifling makes $n = 30$, we find $Q = \frac{WD \,.\, 2\pi}{16\times 30}$, or 0·0131 WD.

Thus the couple varies as the fourth power of the linear dimensions of the projectile and is independent of its velocity. This would be true for the whole length of the trajectory if ω remained equal to $2\pi v/nD$; but there is good reason to believe that the actual value of ω, which is only affected by air friction, does not decrease nearly as rapidly as v, whose variation depends on air resistance.

The couple set up by the air friction and wake at the head and tail of the shot is presumably proportional at any instant to ω^2, to the angle (ψ) which the axis of the projectile makes with the direction of motion, and to some at present unknown function of the velocity $F(v)$. So that $\omega^2\psi F(v)$ must be equal to $MK^2\omega\Omega$ if the axis of the projectile is to keep at a nearly constant angular distance from the direction of motion, or

$$\psi = MK^2\Omega/\omega F(v) = MK^2 g/\omega v F(v). \qquad (4)$$

Hence as v diminishes, ψ will have to increase, and it is only while moderate increments of ψ, such as will leave ψ itself small, satisfy the conditions in (4) that the motion is approximately steady.

The upward force (f) due to the inclination of the axis of the projectile to the tangent of the trajectory may be taken as proportional to ψ/R, R being the air resistance, and ψ' the projection of ψ on the vertical plane. So that the effective downward acceleration of the shot is $g-f$.

If the shot is fired at an inclination of α to the horizon, it would, in the absence of any force other than air resistance, have travelled in the time t a distance s $(= v't+\frac{v_0}{a}(1-e^{-at}))$, and have attained a height equal to $s\sin\alpha$ above the ground. If the acceleration $g-f$ be resolved parallel and perpendicular to the original direction of motion, the distance traversed in time t will be $s-\sin\alpha\iint(g-f)\,dt$ parallel to the direction defined by the elevation and $\cos\alpha\iint(g-f)\,dt$ normal to that direction.

The result is that at t the projectile will be exactly under the position s, having fallen through a distance $\iint(g-f)\,dt$.

If we put $\iint(g-f) = \frac{1}{2}g't^2$ (g' may be called the apparent value of gravity) and make $s\sin\alpha = \frac{1}{2}g't^2$, then $s\cos\alpha$ (or, if α is small, s) is the range for elevation α.

By the analysis of a large number of range tables, I find that g' is well represented by the formula

$$g' = g-c\,(u_0-u),$$

where, if the units are feet and seconds, $c = 8{\cdot}4/u_0$.

The values thus found are accurate enough for range-table purposes, but experiments are much wanted to determine the real relations between f, ψ,* ω, and α.

In the foregoing formulæ no account is taken of the variation of density of the air at different levels. Many projectiles, however, reach heights at which the variation of density cannot be neglected. A comparatively simple method of correcting the range for varying density is as follows:—The height (h) which projectiles reach with "direct fire" is small compared to H, the height of the homogeneous atmosphere. Thus the (a) of the foregoing formulæ becomes, for a shot at altitude h, $a\,(1-h/\mathrm{H})$.

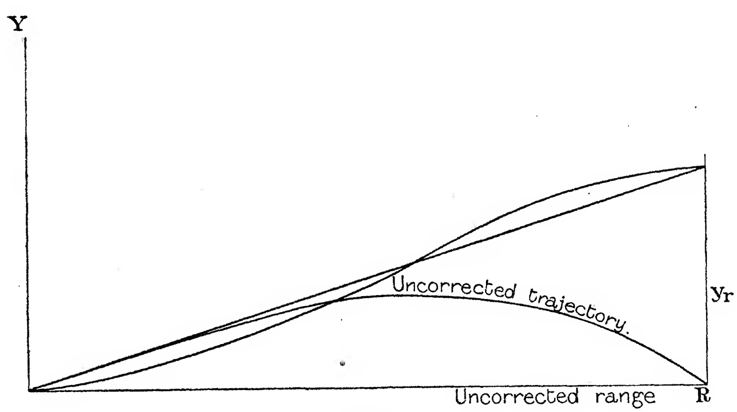

Fig. 6.

* ψ will vary with the shape of the projectile.

In the expression for v, namely $v = v' + u_0 e^{-at}$, suppose t constant and differentiated with respect to a,

$$du/da = -tu_0 e^{-at} = -tv.$$

Thus $$v + \delta v = v' + u(1 - t\delta a), \quad \text{but} \quad \delta a = -ah/\mathrm{H};$$

therefore $$v + \frac{dv}{dn}\delta h = v' + v\left(1 + at\frac{h}{\mathrm{H}}\right).$$

Now, with the range ($s \cos a$) as abscissa, draw a curve with ordinate y such that

$$\frac{d\mathrm{R}}{(d\mathrm{R}^2 + dy^2)^{\frac{1}{2}}} = \frac{v' + u}{v' + u(1 + ath/\mathrm{H})}.$$

Then the length of this curve is the range corrected for variation of density.*

The form of the curve is shown in the diagram, fig. 6, and an approximation to its length is $(\mathrm{R}^2 + y_v{}^2)^{\frac{1}{2}}$.

It will be found that for the larger guns the angle $\sin^{-1}(y_\mathrm{R})/\mathrm{R}$ is of the same order of magnitude as a, that the corrected range is nearly equal to s; and it is for this reason that the formula range $= s$ (instead of $s \cos a$) is practically correct, even when the angle of elevation is as large as 9°. For small arms the range given by the formula would be slightly in excess of the true range with elevation as large as this.

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* The height which the projectile reaches in time t varies with the density of the air, but for the purpose of correcting the range it is sufficient to take h as what it would have been with air of constant density.